Pronormality of Hall subgroups in finite simple groups[1]

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Abstract

We prove that Hall subgroups of finite simple groups are pronormal. Thus we obtain an affirmative answer to Problem 17.45(a) of "Kourovka notebook".

Introduction

According to definition of P.Hall, a subgroup H of a group G is called *pronormal*, if for every $g \in G$ subgroups H and H^g are conjugate in $\langle H, H^g \rangle$. Classical examples of pronormal subgroups are:

- normal subgroups;
- maximal subgroups;
- Sylow subgroups of finite groups;
- Carter subgroups (i. e., nilpotent selfnormalizing subgroups) of finite solvable groups;
- Hall subgroups (i. e. subgroups whose order and index are coprime) of finite solvable groups.

The pronormality of subgroups in last three cases follows from conjugacy of Sylow, Carter, and Hall subgroups in finite groups in corresponding classes. In [1, Theorem 9.2] the first author proved that Carter subgroups in finite groups are conjugate. As a corollary it follows that Carter subgroups of finite groups are pronormal.

In contrast with Carter subgroups, Hall subgroups in finite groups can be non-conjugate. The goal of the authors is to find classes of finite groups with pronormal Hall subgroups. In the present paper the following result is obtained.

Theorem 1. *Hall subgroups of finite simple groups are pronormal.*

The theorem gives an affirmative answer to Problem 17.45(a) from the "Kourovka notebook" [2], and it is announced by the authors in [3, Theorem 7.9]. This result is supposed to use for studying the problem, whether C_π is inherited by overgroups of π-Hall subgroups [2, Problem 17.44(a); 4, Conjecture 3; 5, Problems 2, 3] (all definitions are given below).

1 Notation, conventions, and preliminary results

Notation in the paper are standard.

If G is a finite group, H is its subgroup, and x is an element of G, then by $Z(G)$, $O_\infty(G)$, $N_G(H)$, $C_G(H)$, and $C_G(x)$ we denote the center of G, the solvable radical of G, the normalizer of H in G, the centralizer of H in G, and the centralizer of x in G respectively. Given groups A and B by $A \times B$ and $A \circ B$ we denote the direct product and a central product respectively. If A and B are subgroups of G, then by $\langle A, B \rangle$ and $[A, B]$ the subgroup generated by $A \cup B$ and mutual commutant of A and B are denoted.

We often use the notations from [6]. In particular, by $A : B$, $A \cdot B$, and $A . B$ we denote a split, a nonsplit, and an arbitrary extensions of A by B respectively. Given group G and a subgroup S of the symmetric group Sym_n we denote permutation wreath product of G and S by $G \wr S$ (here n and the embedding of S into Sym_n assumed to be known).

We write $H \operatorname{prn} G$ if H is a pronormal subgroup of G.

Throughout π denotes a set of primes. A natural number n with $\pi(n) \subseteq \pi$, is called a π-*number*, while a group G with $\pi(G) \subseteq \pi$ is called a π-*group*. Symbol n_π is used for the maximal π-number dividing n. A subgroup H of G is called a π-*Hall subgroup*, if $\pi(H) \subseteq \pi$ and $\pi(|G : H|) \subseteq \pi'$. The set of all π-Hall subgroups of G we denote by $\mathrm{Hall}_\pi(G)$. A Hall subgroup is π-Hall subgroup for some π.

According to [7] we say that G *satisfies* E_π (or briefly $G \in E_\pi$), if G possesses a π-Hall subgroup. If, moreover, every two π-Hall subgroups are conjugate, then we say that G *satisfies* C_π ($G \in C_\pi$). If, in addition, each π-subgroup of G is included in a π-Hall subgroup, then we say that G *satisfies* D_π ($G \in D_\pi$). A group satisfying E_π (C_π, D_π) we also call an E_π- (respectively a C_π-, a D_π-) *group*.

A finite group possessing a (sub)normal series such that all factors of the series are either π- or π'-groups is called π-*separable*.

Lemma 2. [7, Lemma 1] *Let A be a normal subgroup of a finite group G. If $G \in E_\pi$ and $H \in \mathrm{Hall}_\pi(G)$, then $A, G/A \in E_\pi$, moreover $H \cap A \in \mathrm{Hall}_\pi(A)$ and $HA/A \in \mathrm{Hall}_\pi(G/A)$.*

Lemma 3. [8; 7, Corollary D5.2] *A π-separable group satisfies D_π.*

Lemma 4. *Let H be a subgroup of G, $g \in G$, $y \in \langle H, H^g \rangle$. If subgroups H^y and H^g are conjugate in $\langle H^y, H^g \rangle$, then H and H^g are conjugate in $\langle H, H^g \rangle$.*

Proof. Let $z \in \langle H^y, H^g \rangle$, and $H^{yz} = H^g$. Then $z \in \langle H, H^g \rangle$ since $\langle H^y, H^g \rangle \leqslant \langle H, H^g \rangle$. Put $x = yz$. Then $x \in \langle H, H^g \rangle$ and $H^x = H^g$. $\qquad\square$

Lemma 5. *Let H be a subgroup of a finite G. Assume that H includes a pronormal (for example, a Sylow) subgroup S of G. Then the following statements are equivalent:*
    (1) $H \operatorname{prn} G$;
    (2) H *and* H^g *are conjugate in* $\langle H, H^g \rangle$ *for each* $g \in N_G(S)$.

Proof. Clearly (1) $\Rightarrow$ (2). We prove that (2) $\Rightarrow$ (1). Assume that statement (2) holds. Choose arbitrary $g \in G$. Notice that $S, S^g \leqslant \langle H, H^g \rangle$. Since S is pronormal, there exists $y \in \langle S, S^g \rangle \leqslant \langle H, H^g \rangle$ such that the equality $S^{gy} = S$ holds. In particular, $gy \in N_G(S)$. In view of (2), subgroups H and H^{gy} are conjugate in $\langle H, H^{gy} \rangle$. Then $H^{y^{-1}}$ and H^g are conjugate in $\langle H^{y^{-1}}, H^g \rangle$. Now H and H^g are conjugate in $\langle H, H^g \rangle$ by Lemma 4. $\qquad\square$

Lemma 6. *Let $^- : G \to G_1$ be a homomorphism of groups, $H \leq G$. If $H \operatorname{prn} G$, then $\overline{H} \operatorname{prn} \overline{G}$.*

Proof. Clear. $\qquad\square$

Lemma 7. *Let G be a finite group and $G_1, \dots, G_n$ be normal subgroups of G such that $[G_i, G_j] = 1$ for $i \neq j$ and $G = G_1 \dots G_n$. Assume that for each $i = 1, \dots, n$ a pronormal subgroup H_i of G_i is chosen, and $H = \langle H_1, \dots, H_n \rangle$. Then $H \operatorname{prn} G$.*

Proof. Choose arbitrary $g \in G$. Then $g = g_1 \dots g_n$ for some $g_1 \in G_1, \dots, g_n \in G_n$. Since H_i is pronormal in G_i for each $i = 1, \dots, n$, there exist $x_i \in \langle H_i, H_i^{g_i} \rangle$ such that $H_i^{x_i} = H_i^{g_i}$. Since $[G_i, G_j] = 1$ for $i \neq j$, we have $H_i^g = H_i^{g_i}$ for each $i = 1, \dots, n$. The same arguments imply $H_i^{x_i} = H_i^x$, where $x = x_1 \dots x_n$. Clearly

$$x \in \left\langle H_i, H_i^{g_i} \mid i = 1, \dots, n \right\rangle = \left\langle H_i, H_i^g \mid i = 1, \dots, n \right\rangle = \langle H, H^g \rangle.$$

Further,

$$H^g = \left\langle H_i^g \mid i = 1, \ldots, n \right\rangle = \left\langle H_i^{g_i} \mid i = 1, \ldots, n \right\rangle = \left\langle H_i^{x_i} \mid i = 1, \ldots, n \right\rangle$$
$$= \left\langle H_i^x \mid i = 1, \ldots, n \right\rangle = H^x. \quad (1)$$

$\square$

Lemma 8. *Let G be a finite group, $H \in \mathrm{Hall}_\pi(G)$, $A \trianglelefteq G$, and $G = HA$. If $(H \cap A) \operatorname{prn} A$, then $H \operatorname{prn} G$.*

Proof. By Lemma 2, $H \cap A$ is a π-Hall subgroup of A. Let $(H \cap A) \operatorname{prn} A$. Choose arbitrary $g \in G$ and show that $H^x = H^g$ for some $x \in \langle H, H^g \rangle$.

Since $G = HA$, there exist $h \in H$ and $a \in A$ such that $g = ha$. Since $(H \cap A) \operatorname{prn} A$, there exists $y \in \langle H \cap A, H^a \cap A \rangle$ such that $H^y \cap A = H^a \cap A$. In view of

$$y \in \langle H \cap A, H^a \cap A \rangle \leq \langle H, H^a \rangle = \langle H, H^{ha} \rangle = \langle H, H^g \rangle,$$

and Lemma 4 we need to consider the case $H = H^y$. In particular,

$$H \cap A = H^a \cap A = H^g \cap A.$$

Now H, H^g, and g are included in $N_G(H \cap A)$. Since $G = HA$ we have $G = AN_G(H \cap A)$. Notice that

$$N_G(H \cap A)/N_A(H \cap A) = N_G(H \cap A)/(A \cap N_G(H \cap A)) \simeq AN_G(H \cap A)/A = G/A$$

is a π-group. Consider a normal series

$$N_G(H \cap A) \trianglerighteq N_A(H \cap A) \trianglerighteq H \cap A \trianglerighteq 1$$

of $N_G(H \cap A)$. Each factor of the series is either a π- or a π'-group, so $N_G(H \cap A)$ is π-separable. Therefore, the subgroup $\langle H, H^g \rangle$ of $N_G(H \cap A)$ is π-separable as well, and in particular $\langle H, H^g \rangle \in D_\pi$ by Lemma 3. Thus π-Hall subgroups H and H^g are conjugate in $\langle H, H^g \rangle$. $\square$

The next lemma gives a sufficient condition for the treatment of lemma 6 in case when H is a Hall subgroup of G.

Lemma 9. *Let $\mathfrak{X}$ be a class of finite groups close under subgroups such that $\mathfrak{X} \subseteq C_\pi$. Let G be a finite group, $H \in \mathrm{Hall}_\pi(G)$, $A \trianglelefteq G$, and $^- : G \to G/A$ be the natural homomorphism. Assume also that $A \in \mathfrak{X}$. Then $H \operatorname{prn} G$ if and only if $\overline{H} \operatorname{prn} \overline{G}$.*

Proof. The implication $\Rightarrow$ holds by Lemma 6.

We prove $\Leftarrow$. Let $g \in G$. We need to show that $H^x = H^g$ for some $x \in \langle H, H^g \rangle$. Since $\overline{H} \operatorname{prn} \overline{G}$, there exists $y \in \langle H, H^g \rangle$ such that $H^y A = H^g A$. By Lemma 4 we may substitute H by H^y and so we may assume that $HA = H^g A$.

Consider $M = \langle H \cap A, H^g \cap A \rangle$. Since $M \leq A$, $A \in \mathfrak{X}$ and $\mathfrak{X}$ is closed under subgroups, we have $M \in \mathfrak{X} \subseteq C_\pi$. Further $H \cap A, H^g \cap A \in \mathrm{Hall}_\pi(A)$ by Lemma 2, and $M \leqslant A$, so $H \cap A, H^g \cap A \in \mathrm{Hall}_\pi(M)$. Hence $H^a \cap A = H^g \cap A$ for some $a \in M$. Since $M \leq \langle H, H^g \rangle$, by Lemma 4 we may substitute H by H^a, and so we may assume that $H \cap A = H^g \cap A$. In such case $g \in N_G(H \cap A)$ and $H, H^g \leq N_G(H \cap A)$. Since $A \in C_\pi$ by Frattini argument we have $G = AN_G(H \cap A)$. Now

$$N_G(H \cap A)/N_A(H \cap A) = N_G(H \cap A)/(A \cap N_G(H \cap A)) \simeq AN_G(H \cap A)/A = G/A = \overline{G}.$$

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As we noted above $\overline{H} = \overline{H}^{\overline{g}}$, so the isomorphism implies that $HN_A(H \cap A) = H^g N_A(H \cap A)$. Denote the last subgroup by B for brevity. Then B is π-separable and $H, H^g \leq B$. Moreover, $\langle H, H^g \rangle$ is also π-separable as a subgroup of a π-separable group B. In particular, by Lemma 3

$$\langle H, H^g \rangle \in D_\pi \quad \text{and} \quad H, H^g \in \text{Hall}_\pi(\langle H, H^g \rangle),$$

whence H and H^g are conjugate in $\langle H, H^g \rangle$. $\qquad\square$

Let G be a finite group and $1\pi(G) = \{p_1, \ldots, p_n\}$. Following [7] we say that G has a *Sylow series of complexion*[2] $(p_1, \ldots, p_n)$, if G possesses a normal series

$$G = G_0 > G_1 > \cdots > G_n = 1$$

such that each section G_{i-1}/G_i is isomorphic to a Sylow p_i-subgroup of G.

Lemma 10. *Let G be a finite group, H be its Hall subgroup with a Sylow series. Then $H \operatorname{prn} G$.*

Proof. Let $g \in G$. We show that H and H^g are conjugate in $\langle H, H^g \rangle$. By [7, Theorem A1] every two Hall subgroups of a finite groups having Sylow series of the same complexion are conjugate. Since H and H^g are two Hall subgroups of $\langle H, H^g \rangle$ having Sylow series of the same complexion, H and H^g are conjugate in $\langle H, H^g \rangle$. $\qquad\square$

Lemma 11. *Let G be a finite nonabelian simple group, H be its Hall subgroup of order not divisible either by 2 or by 3. Then H has a Sylow series.*

Proof. If 2 does not divide the order of H, the claim is proven in [9, Theorem B]. If 3 does not divide the order of H the claim follows from [10, Lemma 5.1, Theorem 5.2]. $\qquad\square$

The symmetric group and the alternating group of degree n we denote by Sym_n and Alt_n respectively.

A finite field containing q elements, is denoted by $\mathbb{F}_q$.

Given odd number q define $\varepsilon(q) = (-1)^{(q-1)/2}$, i. e., $\varepsilon(q) = 1$, if $q - 1$ is divisible b 4, and $\varepsilon(q) = -1$ otherwise. Without additional explanations we use symbols ε, η to denote either an element from $\{+1, -1\}$ or the sign of the element.

Given group of Lie type the order of the base field is always denoted by q (see [1], for example), while its characteristic is denoted by p. Given matrix group G the reduction modulo scalars is denoted by PG.

Our notation for classical groups agrees with that of [11]. We recall special notation, that we often use:

$\text{GL}_n^+(q) = \text{GL}_n(q)$ is a general linear group of degree n over $\mathbb{F}_q$;

$\text{SL}_n^+(q) = \text{SL}_n(q)$ is a special linear group of degree n over $\mathbb{F}_q$;

$\text{PGL}_n^+(q) = \text{PGL}_n(q)$ is a projective general linear group of degree n over $\mathbb{F}_q$;

$\text{PSL}_n^+(q) = \text{PSL}_n(q)$ is a projective special linear group of degree n over $\mathbb{F}_q$;

$\text{GL}_n^-(q) = \text{GU}_n(q)$ is a general unitary group of degree n over $\mathbb{F}_{q^2}$;

$\text{SL}_n^-(q) = \text{SU}_n(q)$ is a special unitary group of degree n over $\mathbb{F}_{q^2}$;

$\text{PSL}_n^-(q) = \text{PSU}_n(q)$ is a projective special unitary group of degree n over $\mathbb{F}_{q^2}$;

$\text{PGL}_n^-(q) = \text{PGU}_n(q)$ is a projective general unitary group of degree n over $\mathbb{F}_{q^2}$;

$\text{Sp}_n(q)$ is a simplectic group of degree n over $\mathbb{F}_q$;

$\text{PSp}_n(q)$ is a projective simplectic group of degree n over $\mathbb{F}_q$.

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[2]Parentheses in the notation $(p_1, \ldots, p_n)$ are used for an ordered set, apart from braces. For example, the symmetric group Sym_3 has a Sylow series of complexity $(2, 3)$, while the alternating group Alt_4 has a Sylow series of complexity $(3, 2)$.

Necessary facts about properties and structure of finite groups of Lie type can be found in [12–15], properties and structure of linear algebraic groups can be found in [12], results concerning the connection between groups of Lie type and linear algebraic groups can be found in [13–14]. Also in [13–14] the definitions of Borel an Cartan subgroups, a parabolic subgroup, and a maximal torus in a finite group of Lie type can be found.

We denote groups $E_6(q)$ and $^2E_6(q)$ by $E_6^+(q)$ and $E_6^-(q)$ respectively.

A *Frobenius map* of an algebraic group $\overline{G}$ is a surjective endomorphism $\sigma : \overline{G} \to \overline{G}$ such that the set of its stable points $\overline{G}_\sigma$ is finite. Each simple group of Lie type of a finite field F of characteristic p is known to coincide with $O^{p'}(\overline{G}_\sigma)$ for an appropriate linear algebraic group $\overline{G}$ over the algebraci closure of F and a Frobenius map σ, where $O^{p'}(\overline{G}_\sigma)$ is a subgroup of $\overline{G}_\sigma$ generated by all p-elements.

Let $\overline{R}$ be a closed σ-stable subgroup of an algebraic group $\overline{G}$ for a Frobenius map σ of $\overline{G}$. Consider subgroups $R = G \cap \overline{R}$ and $N(G, R) = G \cap N_{\overline{G}}(\overline{R})$, where $G = O^{p'}(\overline{G}_\sigma)$. Notice that $N(G, R) \leq N_G(R)$ and $N(G, R) \neq N_G(R)$ in general.

Lemma 12. [16, Corollary of Theorems 1–3] *Let G be a finite simple nonabelian group and $S \in \mathrm{Syl}_2(G)$. Then $N_G(S) = S$, except the following cases:*

(1) $G \simeq J_2$, J_3, Suz or HN and $|N_G(S) : S| = 3$;

(2) $G \simeq {}^2G_2(q)$ or J_1 and $N_G(S) \simeq 2^3.7.3 < \mathrm{Hol}(2^3)$;

(3) G *is a group of Lie type over a field of characteristic* 2 *and* $N_G(S)$ *is a Borel subgroup of* G;

(4) $G \simeq \mathrm{PSL}_2(q)$, *where* $3 < q \equiv \pm 3 \pmod 8$ *and* $N_G(S) \simeq \mathrm{Alt}_4$;

(5) $G \simeq E_6^\eta(q)$, $\eta = \pm$, q *is odd and* $N_G(S) = S \times C$, *where* C *is a nontrivial cyclic group of order* $(q - \eta)_{2'}/(q - \eta, 3)_{2'}$;

(6) $G \simeq \mathrm{PSp}_{2m}(q)$, $m \geqslant 2$, $q \equiv \pm 3 \pmod 8$, *the factor group* $N_G(S)/S$ *is isomorphic to an elementary abelian* 3-group of order 3^t *and* t *can be found from the* 2-adic decomposition

$$m = 2^{s_1} + \cdots + 2^{s_t},$$

where $s_1 > \cdots > s_t \geqslant 0$;

(7) $G \simeq \mathrm{PSL}_n^\eta(q)$, $n \geqslant 3$, $\eta = \pm$, q *is odd*,

$$N_G(S) \simeq S \times C_1 \times \cdots \times C_{t-1},$$

t can be found from a 2-adic decomposition

$$n = 2^{s_1} + \cdots + 2^{s_t},$$

where $s_1 > \cdots > s_t \geq 0$, and $C_1, \ldots, C_{t-2}, C_{t-1}$ are cyclic groups of orders $(q - \eta)_{2'}, \ldots, (q - \eta)_{2'}$, $(q - \eta)_{2'}/(q - \eta, n)_{2'}$ respectively.

Lemma 13. [7, Theorem A4; 17] *Let $2, 3 \in \pi$. Then the list of all cases, when Sym_n possesses a proper π-Hall subgroup is given in Table 1. In particular, each proper π-Hall subgroup of Sym_n is maximal in Sym_n.*

Table 1: π-Hall subgroups in symmetric groups

n	$\pi \cap \pi(\mathrm{Sym}_n)$	$H \in \mathrm{Hall}_\pi(\mathrm{Sym}_n)$
prime	$\pi((n-1)!)$	Sym_{n-1}
7	$\{2, 3\}$	$\mathrm{Sym}_3 \times \mathrm{Sym}_4$
8	$\{2, 3\}$	$\mathrm{Sym}_4 \wr \mathrm{Sym}_2$

Lemma 14. [18, Theorem 4.1] *Let G be either one of 26 sporadic groups, or the Tits group. Assume that π contains both 2 and 3. Then G possesses a proper π-Hall subgroup H if and only if one of the conditions on G and $\pi \cap \pi(G)$ from Table. 2 holds. In the table the structure of H is also given.*

Table 2: π-Hall subgroups in sporadic groups, case $2, 3 \in \pi$

G	$\pi \cap \pi(G)$	Structure H
M_{11}	$\{2, 3\}$	$3^2 : Q_8 . 2$
	$\{2, 3, 5\}$	$\mathrm{Alt}_6 . 2$
M_{22}	$\{2, 3, 5\}$	$2^4 : \mathrm{Alt}_6$
M_{23}	$\{2, 3\}$	$2^4 : (3 \times A_4) : 2$
	$\{2, 3, 5\}$	$2^4 : \mathrm{Alt}_6$
	$\{2, 3, 5\}$	$2^4 : (3 \times \mathrm{Alt}_5) : 2$
	$\{2, 3, 5, 7\}$	$\mathrm{L}_3(4) : 2_2$
	$\{2, 3, 5, 7\}$	$2^4 : \mathrm{Alt}_7$
	$\{2, 3, 5, 7, 11\}$	M_{22}
M_{24}	$\{2, 3, 5\}$	$2^6 : 3 \, \dot{} \, \mathrm{Sym}_6$
J_1	$\{2, 3\}$	$2 \times \mathrm{Alt}_4$
	$\{2, 3, 5\}$	$2 \times A_5$
	$\{2, 3, 7\}$	$2^3 : 7 : 3$
J_4	$\{2, 3, 5\}$	$2^{11} : (2^6 : 3 \, \dot{} \, \mathrm{Sym}_6)$

Lemma 15. [19, Theorem 3.3] *Let G be a finite group of Lie type over a field of characteristic $p \in \pi$. If H is a π-Hall subgroup of G, then either H is included in a Borel subgroup, or H is a parabolic subgroup of G.*

Lemma 16. [20, Lemma 3.1] *Let $G \simeq \mathrm{P\,SL}_2(q) \simeq \mathrm{P\,SL}_2^\eta(q) \simeq \mathrm{P\,Sp}_2(q)$, where q is a power of an odd prime p, and set $\varepsilon = \varepsilon(q)$. Assume that $2, 3 \in \pi$, and $p \notin \pi$. Then $G \in E_\pi$ if and only of on of the cases from Table 3 holds.*

Table 3: π-Hall subgroups H of $\mathrm{PSL}_2(q)$, $2, 3 \in \pi$, $p \notin \pi$

$\pi \cap \pi(G)$	H	conditions
$\subseteq \pi(q - \varepsilon)$	$D_{q-\varepsilon}$	—
$\{2, 3\}$	Alt_4	$(q^2 - 1)_{\{2,3\}} = 24$
$\{2, 3\}$	Sym_4	$(q^2 - 1)_{\{2,3\}} = 48$
$\{2, 3, 5\}$	Alt_5	$(q^2 - 1)_{\{2,3,5\}} = 120$

Lemma 17. [20, Lemma 3.2] *Assume that $G = \mathrm{GL}_2^\eta(q)$, where q is a power of a prime p, $\mathrm{P} : G \to G/Z(G) = \mathrm{PGL}_2^\eta(q)$ is the natural homomorphism, and let $\varepsilon = \varepsilon(q)$. Assume also that $2, 3 \in \pi$ and $p \notin \pi$. A subgroup H of G is a π-Hall subgroup if and only if one of the following statements holds:*

(1) $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$, $\mathrm{P}H$ is a π-Hall subgroup of the dihedral group $D_{2(q-\varepsilon)}$ of order $2(q - \varepsilon)$ of $\mathrm{P}G$;

(2) $\pi \cap \pi(G) = \{2, 3\}$, $(q^2 - 1)_{\{2,3\}} = 24$, $\mathrm{P}H \simeq \mathrm{Sym}_4$.

Moreover every two π-Hall subgroups of G, satisfying to the same statement (1) or (2) are conjugate.

Lemma 18. [20, Lemma 4.3] *Let $G^* = \mathrm{SL}_n^\eta(q)$ be a special linear or unitary group with*

the base field $\mathbb{F}_q$ of characteristic p, and let $n \geq 2$. Assume that $2, 3 \in \pi$ and $p \notin \pi$. Suppose that $G^* \in E_\pi$ and H^* is a π-Hall subgroup of G^*. Then for G^*, H^* and π one of the following statements holds.

(1) $n = 2$ and for groups $G = G^*/Z(G^*)$ and $H = H^*Z(G^*)/Z(G^*)$ the conditions from Table 3 holds.

(2) Either $q \equiv \eta \pmod{12}$, or $n = 3$ and $q \equiv \eta \pmod 4$; Sym_n satisfies E_π, $\pi \cap \pi(G^*) \subseteq \pi(q - \eta) \cup \pi(n!)$ and if $r \in (\pi \cap \pi(n!)) \setminus \pi(q - \eta)$, then $|G^*|_r = |\mathrm{Sym}_n|_r$; H^* is included in

$$M = L \cap G^* \simeq Z^{n-1} \cdot \mathrm{Sym}_n,$$

where $L = Z \wr \mathrm{Sym}_n \leq \mathrm{GL}_n^\eta(q)$ and $Z = \mathrm{GL}_1^\eta(q)$ is a cyclic group of order $q - \eta$.

(3) $n = 2m + k$, where $k \in \{0, 1\}$, $m \geqslant 1$, $q \equiv -\eta \pmod 3$, $\pi \cap \pi(G^*) \subseteq \pi(q^2 - 1)$, both Sym_m and $\mathrm{GL}_2^\eta(q)$ satisfy E_π [3]; H^* is included in

$$M = L \cap G^* \simeq \big(\underbrace{\mathrm{GL}_2^\eta(q) \circ \cdots \circ \mathrm{GL}_2^\eta(q)}_{m \text{ times}}\big) \cdot \mathrm{Sym}_m \circ Z,$$

where $L = \mathrm{GL}_2^\eta(q) \wr \mathrm{Sym}_m \times Z \leqslant \mathrm{GL}_n(q)$ and Z is a cyclic group of order $q - \eta$ if $k = 1$, and $Z = 1$ if $k = 0$. A subgroup H^* acting by conjugation on the set of factors of type $\mathrm{GL}_2^\eta(q)$ in the central product

$$\underbrace{\mathrm{GL}_2^\eta(q) \circ \cdots \circ \mathrm{GL}_2^\eta(q)}_{m \text{ times}},$$

has at most two orbits. The intersection of H^* with each factor $\mathrm{GL}_2^\eta(q)$ in (1) is a π-Hall subgroup of $\mathrm{GL}_2^\eta(q)$. All intersections of H^* with factors from the same orbit satisfy to the same statement (1) or (2) in Lemma 17.

(4) $n = 4$, $\pi \cap \pi(G^*) = \{2, 3, 5\}$, $q \equiv 5\eta \pmod 8$, $(q + \eta)_3 = 3$, $(q^2 + 1)_5 = 5$ and $H^* \simeq 4 \cdot 2^4 \cdot \mathrm{Alt}_6$.

(5) $n = 11$, $\pi \cap \pi(G^*) = \{2, 3\}$, $(q^2 - 1)_{\{2,3\}} = 24$, $q \equiv -\eta \pmod 3$, $q \equiv \eta \pmod 4$, H^* is included in $M = L \cap G^*$, where L is a subgroup of G^* of type $\big(\big(\mathrm{GL}_2^\eta(q) \wr \mathrm{Sym}_4\big) \perp \big(\mathrm{GL}_1^\eta(q) \wr \mathrm{Sym}_3\big)\big)$ and

$$H^* = \big(\big(\big(Z \circ 2 \cdot \mathrm{Sym}_4\big) \wr \mathrm{Sym}_4\big) \times \big(Z \wr \mathrm{Sym}_3\big)\big) \cap G,$$

where Z is a Sylow 2-subgroup of a cyclic group of order $q - \eta$.

Lemma 19. [20, Lemma 4.4] *Let* $G^* = \mathrm{Sp}_{2n}(q)$ *be a simplectic group over a field* $\mathbb{F}_q$ *of characteristic* p. *Assume that* $2, 3 \in \pi$ *and* $p \notin \pi$. *Suppose that* $G^* \in E_\pi$ *and* $H^* \in \mathrm{Hall}_\pi(G)$. *Then both* Sym_n *and* $\mathrm{SL}_2(q)$ *satisfy* E_π *and* $\pi \cap \pi(G^*) \subseteq \pi(q^2 - 1)$. *Moreover* H^* *is a* π-Hall *subgroup of*

$$M = \mathrm{Sp}_2(q) \wr \mathrm{Sym}_n \simeq \big(\underbrace{\mathrm{SL}_2(q) \times \cdots \times \mathrm{SL}_2(q)}_{n \text{ times}}\big) : \mathrm{Sym}_n \leqslant G^*.$$

Lemma 20. [20, Lemma 7.3] *Let* $G = E_6^\eta(q)$, *where* q *is a power of a prime* p, *and* $\varepsilon = \varepsilon(q)$. *Assume that* $2, 3 \in \pi$ *and* $p \notin \pi$. *Suppose that* G *possesses a* π-Hall *subgroup* H. *Then* $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$ *and one of the following statements holds:*

(1) $\eta = \varepsilon$, $5 \in \pi$ *and* H *is a* π-Hall *subgroup of*

$$M = ((q - \eta)^6 . W(E_6))/(3, q - \eta);$$

(2) $\eta = -\varepsilon$ *and* H *is a* π-Hall *subgroup of*

$$M = (q^2 - 1)^2 (q + \eta)^2 . W(F_4).$$

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[3] By Lemma 16 conditions $\mathrm{GL}_2^\eta(q) \in E_\pi$ and $q \equiv -\eta \pmod 3$ mean that $q \equiv -\eta \pmod r$ for all odd primes $r \in \pi(q^2 - 1) \cap \pi$.

2 Proof of Theorem 1

Let G be a finite simple group and $H \in \mathrm{Hall}_\pi(G)$. We show that $H \, \mathrm{prn} \, G$, and thus we prove Theorem 1. By Lemmas 10 and 11 we may assume that $2, 3 \in \pi$. Let $S \in \mathrm{Syl}_2(H) \subseteq \mathrm{Syl}_2(G)$ and $g \in N_G(S)$ be arbitrary. By Lemma 5 it is enough to prove that H and H^g are conjugate in $\langle H, H^g \rangle$. If $N_G(S) = S$, then this statement is true: $g \in N_G(S) = S \leq H$, so $H^g = H$. Therefore we may assume that one of the exceptional cases (1)–(7) from Lemma 12 holds, and H is a proper π-Hall subgroup of G.

We consider cases (1)–(7) from Lemma 12, proving a series of auxiliary lemmas. In order to unify the notation in Lemmas with already introduced notation we say that $(\star)$ *holds*, if

(a) G is a finite simple group;
(b) $2, 3 \in \pi$;
(c) $H \in \mathrm{Hall}_\pi(G)$ and $H < G$;
(d) $S \in \mathrm{Syl}_2(H) \subseteq \mathrm{Syl}_2(G)$;
(e) $g \in N_G(S)$.

The following lemma follows from Lemma 14 immediately.

Lemma 21. *Assume that $(\star)$ holds. If $G \simeq J_2, J_3, Suz$ or HN, then G does not possesses proper π-Hall subgroups.*

Thus if case (1) of Lemma 12 holds, then by Lemma 5 $H \, \mathrm{prn} \, G$.

Lemma 22. *Assume that $(\star)$ holds. Then the following statements hold.*
(1) If $G \simeq {}^2G_2(q)$, then G does not possesses proper π-Hall subgroup.
(2) If $G \simeq J_1$, then one of the following cases holds:
  (a) $H \simeq 2 \times \mathrm{Alt}_4$ and H possesses a Sylow tower;
  (b) $H \simeq 2^3 : 7 : 3$ and H possesses a Sylow tower;
  (c) $H \simeq 2 \times \mathrm{Alt}_5$ and H is maximal in G.
(3) If $G \simeq J_1$, then H is conjugate with H^g by an element from $\langle H, H^g \rangle$.

Proof. Statement (1) follows from [19, Theorem 1.2], since $3 \in \pi$ and 3 is the characteristic of the base field for ${}^2G_2(q)$. Lemma 14 implies the structure of H in statement (2), moreover it is clear that in cases (a) and (b) the subgroup has a Sylow series, In case (c) H is maximal in view of [6]. Statement (3) follows from (2), Lemma 10, and pronormality of maximal subgroups. □

Thus Lemmas 5 and 22 imply that $H \, \mathrm{prn} \, G$, if statement (2) of Lemma 12 holds.

Lemma 23. *Assume that $(\star)$ holds, and G is a group of Lie type over a field of characteristic 2. Then S is a maximal unipotent subgroup, $N_G(S)$ is a Borel subgroup of G and one of the following statements holds:*
(1) H is included in a Borel subgroup and has a Sylow series;
(2) H is parabolic and includes $N_G(S)$.
In both cases H is conjugate with H^g by an element from $\langle H, H^g \rangle$.

Proof. In view of Lemma 15, the structure of Borel subgroups and the fact that every parabolic subgroup includes a Borel subgroup we obtain that either (1) or (2) holds. By using Lemma 10 we conclude that H is conjugate with H^g by an element from $\langle H, H^g \rangle$, if statement (1) holds. If statement (2) holds the final conclusion is evident, since $g \in H$. □

Thus if statement (3) of Lemma 12 holds, then $H \, \mathrm{prn} \, G$.

Lemma 24. *Assume that* $(\star)$ *holds,* $G = \mathrm{PSL}_2(q)$, $q \equiv \pm 3 \pmod 8$, *and* $q > 3$. *Then one of the following statements holds:*

(1) H *is a* π-*Hall subgroup in a dihedral group of order* $q - \varepsilon$, *where* $\varepsilon = \varepsilon(q) = (-1)^{(q-1)/2}$, *and it has a Sylow series;*

(2) $H \simeq \mathrm{Alt}_4$ *and* H *has a Sylow sereis;*

(3) $H \simeq \mathrm{Alt}_5$ *and* H *includes* $N_G(S) \simeq \mathrm{Alt}_4$, *in particular,* $H^g = H$.

In any case H *is conjugate with* H^g *by an element from* $\langle H, H^g \rangle$.

Proof. Conditions $q \equiv \pm 3 \pmod 8$ and $q > 3$, and Lemma 16 imply the structure of H. Moreover, if either H is included in a dihedral subgroup, or $H \simeq \mathrm{Alt}_4$, then it clearly has a Sylow series. Assume that $H \simeq \mathrm{Alt}_5$. Then $\mathrm{Alt}_4 = N_H(S) \leq N_G(S) \simeq \mathrm{Alt}_4$, so $N_H(S) = N_G(S)$. Using statements (1)–(3) and Lemma 10 we obtain the final conclusion. $\square$

Thus we have shown that if statement (4) of Lemma 12 holds, then $H \operatorname{prn} G$.

Lemma 24 implies also the following statement that is extensively used for consideration of items (6) and (7) in Lemma 12.

Lemma 25. *Let* $2, 3 \in \pi$, q *be a power of an odd prime* $p \notin \pi$,

$$G^* \in \left\{ \mathrm{PSL}_2(q), \ \mathrm{PGL}_2^\eta(q), \ \mathrm{SL}_2(q), \ \mathrm{GL}_2^\eta(q) \right\}$$

and $H^* \in \mathrm{Hall}_\pi(G^*)$. *Then* $H^* \operatorname{prn} G^*$.

Proof. If $G^* = \mathrm{PSL}_2(q)$ and $S^* \in \mathrm{Syl}_2(H^*) \subseteq \mathrm{Syl}_2(G^*)$, then by Lemma 12 either $N_{G^*}(S^*) = S^*$ or G^* satisfies the conditions of Lemma 24. In both cases H^* is pronormal.

Now let $G^* = \mathrm{PGL}_2^\eta(q)$ and $A^* = \mathrm{PSL}_2^\eta(q) \simeq \mathrm{PSL}_2(q)$ be a normal subgroup of index 2 in G^*. As we have already shown, $H^* \cap A^* \operatorname{prn} A^*$ and $G^* = A^* H^*$. Using Lemma 8 we conclude that $H^* \operatorname{prn} G^*$.

Assume finally that G^* is isomorphic to either $\mathrm{SL}_2(q)$ or $\mathrm{GL}_2^\eta(q)$. Choose in Lemma 9 the class of all 2-groups as $\mathfrak{X}$. Then this lemma and the above arguments imply $H^* \operatorname{prn} G^*$. $\square$

Consider statement (5) in Lemma 12.

Lemma 26. *Assume that* $(\star)$ *holds and* $G = E_6^\eta(q)$, *where* q *is a prime of* $p \notin \pi$. *Denote* $\varepsilon(q)$ *by* ε. *Then*

(1) G *includes an* S-*invariant maximal torus* T *such that*

$$|T| = \begin{cases} (q - \varepsilon)^6/(3, q - \varepsilon), & \eta = \varepsilon, \\ (q - \varepsilon)^4 (q + \varepsilon)^2, & \eta = -\varepsilon, \end{cases}$$

moreover $H \leqslant N_G(T)$ *and* $N_G(T)$ *is an extension of* T *by a* π-*group;*

(2) $N_G(T)$ *includes* $N_G(S)$;

(3) *subgroups* H *and* H^g *are conjugate in* $\langle H, H^g \rangle$.

Proof. (1) The existence of S-invariant torus T follows from [12, Theorem 4.10.2]. In view of [10, Lemma 3.10] such torus is unique up to conjugation and $N_G(T) = N(G, T)$. Moreover by [10, Lemma 3.11] the order of T equals $(q - \varepsilon)^6/(3, q - \varepsilon)$, if $\varepsilon = \eta$, and it equals $(q - \varepsilon)^4 (q + \varepsilon)^2$, if $\varepsilon = -\eta$. Since $G \in E_\pi$ and $2, 3 \in \pi$, while $p \notin \pi$, by Lemma 20 we obtain that H lies in $N_G(T)$ for some such torus T and $N_G(T)/T$ is a π-group.

(2) Since $H \leqslant N_G(T)$ and $3 \in \pi$, $N_G(T)$ includes a Sylow 3-subgroup of G. So in follows by [10, Lemma 3.13] that $N_G(S) \leqslant N_G(T)$.

(3) In view of statement (2) of the lemma we remain to prove that $H \operatorname{prn} N_G(T)$. By (1), $N_G(T)$ is an extension of an abelian group T by a π-group, in particular $N_G(T) = HT$. Now, by Lemma 8, $H \operatorname{prn} N_G(T)$. $\square$

Therefore, if statement (5) of Lemma 12 holds, then $H \operatorname{prn} G$.

In the next lemma we consider statement (6) and, partially, statement (7) of Lemma 12. We need to recall the notion of a fundamental subgroup introduced in [21]. We use the notion in simple linear, unitary, and simplectic groups in odd characteristic only, and their central extensions. Recall that if G is one of such groups, X^+ is a long root subgroup of G, and X^- is the opposite root subgroup, then every G-conjugate of $\langle X^+, X^- \rangle \simeq \operatorname{SL}_2(q)$ is called a *fundamental subgroup*. If $S \in \operatorname{Syl}_2(G)$, then by $\operatorname{Fun}_G(S)$ the sen of all fundamental subgroups K of G such that $K \cap S \in \operatorname{Syl}_2(K)$ is denoted. $\operatorname{Fun}_G(S)$ is known to be a maximal by inclusion S-invariant set of pairwise commuting fundamental subgroups of G (see [21]).

Lemma 27. *Assume that* $(\star)$ *holds and G is isomorphic to either* $\operatorname{PSL}_n^\eta(q)$ *or* $\operatorname{PSp}_n(q)$, *where $n > 2$. Let $\Delta = \operatorname{Fun}_G(S)$ and suppose that Δ is H-invariant (i. e. $H \leqslant N_G(\Delta)$ in the notations of* [21]*). Then H and H^g are conjugate in $\langle H, H^g \rangle$.*

Proof. Let $m = [n/2]$. Then $|\Delta| = m$.

In view of [11, Propositions 4.1.4, 4.2.9, and 4.2.10] the stabilizer $N_G(\Delta)$ in G of Δ coincides with the image in G of a subgroup M of either $\operatorname{SL}_n^\eta(q)$ or $\operatorname{Sp}_n(q)$, where M is defined in the following way. If $G = \operatorname{PSL}_n^\eta(q)$, then

$$M = L \cap \operatorname{SL}_n^\eta(q) \simeq \big(\underbrace{\operatorname{GL}_2^\eta(q) \circ \cdots \circ \operatorname{GL}_2^\eta(q)}_{m \text{ times}}\big) \cdot \operatorname{Sym}_m \circ Z,$$

moreover, $L = \operatorname{GL}_2^\eta(q) \wr \operatorname{Sym}_m \times Z \leq \operatorname{GL}_n^\eta(q)$, and Z is a cyclic group of order $q - \eta$ if n is odd, and $Z = 1$ if n is even. If $G = \operatorname{PSp}_n(q)$, then

$$M = \operatorname{Sp}_2(q) \wr \operatorname{Sym}_m \simeq \big(\underbrace{\operatorname{SL}_2(q) \times \cdots \times \operatorname{SL}_2(q)}_{m \text{ times}}\big) : \operatorname{Sym}_m \leq \operatorname{Sp}_n(q).$$

Suppose that the action of $N_G(\Delta)$ on Δ is denoted by

$$\rho : N_G(\Delta) \to \operatorname{Sym}(\Delta) \simeq \operatorname{Sym}_m.$$

By [21, Theorem 2], $N_G(\Delta)^\rho = \operatorname{Sym}(\Delta)$. By Lemma 13 it follows that a π-Hall subgroup H^ρ is either maximal in $\operatorname{Sym}(\Delta)$ or equal to $\operatorname{Sym}(\Delta)$. In particular,

$$H^\rho \operatorname{prn} \operatorname{Sym}(\Delta) \quad \text{and} \quad N_{\operatorname{Sym}(\Delta)}(H^\rho) = H^\rho.$$

Since $N_G(S) \leq N_G(\Delta)$ and $g \in N_G(S)$ there exists an element $y \in \langle H, H^g \rangle$ such that $(H^g)^\rho = (H^y)^\rho$. So

$$(gy^{-1})^\rho \in N_{\operatorname{Sym}(\Delta)}(H^\rho) = H^\rho.$$

Denote by A the kernel of ρ. The structure of $N_G(\Delta)$ implies that if $\overline{} : A \to A/O_\infty(A)$ is a natural homomorphism, then $\overline{A}$ possesses a normal subgroup isomorphic to

$$\underbrace{\operatorname{PSL}_2(q) \times \cdots \times \operatorname{PSL}_2(q)}_{m \text{ times}},$$

and index of the subgroup in $\overline{A}$ is a 2-power. By Lemmas 7–9 (we take the class of solvable groups as $\mathfrak{X}$) and Lemma 25 we conclude that π-Hall subgroups of A are pronormal. Now by π-Hall subgroups of HA are pronormal by Lemma 8. Moreover, $gy^{-1} \in HA$ since $(gy^{-1})^\rho \in H^\rho$. Therefore $H^z = H^{gy^{-1}}$ for some $z \in \langle H, H^{gy^{-1}} \rangle \leq \langle H, H^g \rangle$. Let $x = zy$. Then $H^x = H^g$ and $x \in \langle H, H^g \rangle$. $\qquad \square$

Thus, if either statement (6) of Lemma 12 holds, or statement (7) of the same lemma holds and for the preimage $H^* \leq G^* = \mathrm{SL}^\eta(q)$ of H statement (3) of Lemma 18 holds, then $H \operatorname{prn} G$. Notice also that statements (7) of Lemma 12 and (1) of Lemma 18 hold, then $H \operatorname{prn} G$ by Lemma 25.

The next lemma allows to exclude also the case, when statements (7) of Lemma 12 and (4) of Lemma 18 hold.

Lemma 28. *Let $G = \mathrm{PSL}_4^\eta(q)$, where q is odd. Then $N_G(S) = S$.*

Proof. The claim follows by Lemma 12 since the 2-adic expansion of 4 has only one unit. $\square$

In case, when statements (7) of Lemma 12 and (2) of Lemma 18 hold, H normalizes a maximal torus of order $(q - \eta)^{n-1}/(n, q - \eta)$ of $G = \mathrm{PSL}_n^\eta(q)$. We consider this case as statement (5) of Lemma 12 in the next lemma.

Lemma 29. *Assume that $(\star)$ holds and $G = \mathrm{PSL}_n^\eta(q)$, where q is a power of a prime $p \notin \pi$. Suppose als that $q \equiv \eta \pmod 4$ and there exists a maximal H-invariant torus T of order $(q - \eta)^{n-1}/(n, q - \eta)$. Then*
  (1) $N_G(T) = N(G, T)$;
  (2) $N_G(T)/T \simeq \mathrm{Sym}_n$;
  (3) $N_G(T)$ *includes* $N_G(S)$;
  (4) H *and* H^g *are conjugate in* $\langle H, H^g \rangle$.

Proof. Statement (1) follows from [10, Lemma 3.10], since T is invariant under given Sylow 2-subgroup S of G.

(2) Since the identity $N_G(T) = N(G, T)$ holds, the factor group $N_G(T)/T = N(G, T)/T$ is isomorphic to Sym_n (this factor group is included in the Weyl group of G, which is isomorphic to Sym_n, on the other hand, a subgroup of type $T.\mathrm{Sym}_n$ is included in G ans so in $N_G(T)$).

(3) Since $H \leqslant N_G(T)$ and $3 \in \pi$, $N_G(T)$ includes a Sylow 3-subgroup of G. So, by [10, Lemma 3.13], it follows that $N_G(S) \leqslant N_G(T)$.

(4) In view of statement (3) of the lemma we remain to prove that $H \operatorname{prn} N_G(T)$. By statement (2) of the lemma, $N_G(T)$ is an extension of an abelian group T by Sym_n. Consider the natural epimorphism $\overline{} : N_G(T) \to N_G(T)/T \simeq \mathrm{Sym}_n$. By [20, Lemma 2.1(a)], $\overline{H}$ is a π-Hall subgroup of Sym_n. Since, in view of the condition $2, 3 \in \pi$ and Lemma 13, each π-Hall subgroup is either maximal in Sym_n, or equal to Sym_n, we have $\overline{H} \operatorname{prn} \overline{N_G(T)}$. Taking the class of all abelian groups as $\mathfrak{X}$ in Lemma 9 we obtain that $H \operatorname{prn} N_G(T)$. $\square$

Thus we have considered all possible cases, except the case, when statement (7) of Lemma 12 holds, $G = \mathrm{PSL}_{11}^\eta(q)$, and for the preimage $H^* \leq \mathrm{SL}_{11}^\eta(q)$ of H statement (5) of Lemma 18 holds. In particular the following lemma is true.

Lemma 30. *Let $2, 3 \in \pi$ and q be a power of a prime $p \notin \pi$. Then π-Hall subgroups in $\mathrm{PSL}_n^\eta(q)$, $\mathrm{PGL}_n^\eta(q)$, $\mathrm{SL}_n^\eta(q)$, and $\mathrm{GL}_n^\eta(q)$ for $n \leqslant 4$ and $n = 8$ are pronormal.*

Proof. For $\mathrm{PSL}_n^\eta(q)$ the lemma follows directly from Lemma 18. For $\mathrm{PGL}_n^\eta(q)$ the claim follows from Lemma 8 since

$$\left| \mathrm{PGL}_n^\eta(q) : \mathrm{PSL}_n^\eta(q) \right| = (n, q - \eta)$$

divides n and so it is a π-number. Finally, $\mathrm{SL}_n^\eta(q)$ and $\mathrm{GL}_n^\eta(q)$ are extensions of abelian groups by $\mathrm{PSL}_n^\eta(q)$ and $\mathrm{PGL}_n^\eta(q)$. The assertion of the lemma follows from above arguments and Lemma 9. $\square$

Consider the remaining case. We need

Lemma 31. *Let $G^* = \mathrm{SL}_{11}^\eta(q)$, q be odd, and $S^* \in \mathrm{Syl}_2(G^*)$. Set $\Delta = \mathrm{Fun}_G(S^*)$. Then*
(1) $|\Delta| = 5$ and S^* acting on Δ has exactly two orbits: Γ of order 4 and Γ_0 of order 1;
(2) Γ are Γ_0 $N_{G^*}(S^*)$-invariant;
(3) if Γ' is an S^*-invariant set of pairwise commuting fundamental subgroups of G^* such that $|\Gamma'| = 4$, then $\Gamma' = \Gamma$.

Proof. Denote by ρ the action of $N_{G^*}(\Delta)$ on Δ. According to [21, Theorem 2]

$$N_{G^*}(\Delta)^\rho = \mathrm{Sym}(\Delta) \simeq \mathrm{Sym}_5.$$

S^ρ is a Sylow 2-subgroup of Sym_5 and so it has two orbits on Δ: one orbit of length 4 and another of length 1. This implies statement (1). Statement (2) follows from the fact that S^* and Δ are $N_{G^*}(S^*)$-invariant. Finally, Γ' is included in Δ, since Δ is a unique maximal S^*-invariant set of pairwise commuting fundamental subgroup. So Γ' is a union of some orbits of S^* on Δ and, in view of (1), equals Γ. $\square$

Lemma 32. *Let $G^* = \mathrm{SL}_{11}^\eta(q)$ be a special linear o unitary group and V be its natural module equipped with a trivial or unitary form respectively. Assume that $H^* \in \mathrm{Hall}_\pi(G^*)$, where $\pi \cap \pi(G^*) = \{2, 3\}$, and suppose that H^* is included in a subgroup of type*

$$L = \left(\left(\mathrm{GL}_2^\eta(q) \wr \mathrm{Sym}_4 \right) \times \left(\mathrm{GL}_1^\eta(q) \wr \mathrm{Sym}_3 \right) \right) \cap G^*.$$

Let $S^* \in \mathrm{Syl}_2(H^*) \subseteq \mathrm{Syl}_2(G^*)$ and $g^* \in N_{G^*}(S^*)$. Then
(1) H^* leaves invariant a set $\Gamma' = \{K_1, K_2, K_3, K_4\}$ consisting from pairwise commuting fundamental subgroups;
(2) Γ' is $N_{G^*}(S^*)$-invariant;
(3) if $V_i = [K_i, V]$ and $U = \sum V_i$, then U is invariant under both H^* and $N_{G^*}(S^*)$;
(4) the stabilizer M in G^* of U is a subgroup with pronormal π-Hall subgroups;
(5) $H^* \mathrm{prn} G^*$.

Proof. Consider a subgroup $\left(\mathrm{GL}_2^\eta(q) \wr \mathrm{Sym}_4 \right) \cap G^*$ of $L = \left(\left(\mathrm{GL}_2^\eta(q) \wr \mathrm{Sym}_4 \right) \times \left(\mathrm{GL}_1^\eta(q) \wr \mathrm{Sym}_3 \right) \right) \cap G^*$, and in the base of the wreath product consider distinct normal subgroups K_1, K_2, K_3, K_4 isomorphic to $\mathrm{SL}_2(q)$. Clearly, $K_i \not\leq G^*$ and $K_i \not\leq L$ for all $i = 1, 2, 3, 4$. Moreover, the set $\Gamma' = \{K_1, K_2, K_3, K_4\}$ is L-invariant and so is H^*-invariant. Statement (1) is proven. Statement (2) follows from Lemma 31. Notice that V_i can be considered as the natural module for K_i, therefore $\dim(V_i) = 2$ and $V_i \cap V_j = 0$ for $i \neq j$. In particular, $\dim(U) = 8$. Since Γ' is invariant under both H^* and $N_{G^*}(S^*)$ it follows that the set $\{V_1, V_2, V_3, V_4\}$, and so the subspace U, are also invariant under both H^* and $N_{G^*}(S^*)$. Thus (3) is proven. If $\eta = +$, then the stabilizer M of U is an extension of a p-group by a central product $\mathrm{GL}_8(q) \circ \mathrm{GL}_3(3)$ (see [11, Proposition 4.1.17]), and by Lemmas 30, 7, and 9 we conclude that π-Hall subgroups of M are pronormal. If $\eta = -$, then M is isomorphic to a central product $\mathrm{GU}_8(q) \circ \mathrm{GU}_3(q)$ (see [11, Proposition 4.1.4]), and applying again Lemmas 30 and 7, we obtain statement (4). In view of (3), H^* and every $g^* \in N_{G^*}(S^*)$ are included in M. Now from (4) and Lemma 5 we conclude that $H^* \mathrm{prn} G^*$. $\square$

We continue the proof of the theorem and consider the remaining case. Assume that statement (7) of Lemma 12 holds, $G = \mathrm{PSL}_{11}^\eta(q)$, and for the preimage $H^* \leq \mathrm{SL}_{11}^\eta(q)$ of H statement (5) of Lemma 18 holds. By Lemma 32 we have $H^* \mathrm{prn} \mathrm{SL}_{11}^\eta(q)$. Applying Lemma 6 we conclude the proof of Theorem 1. $\square$

3   Conclusion

In connection with the proof of Theorem 1 we make a small note. The proof is naturally divided into two cases. The first case, when a Hall subgroup H of a finite simple group G has odd order (equivalently, even index). The proof in this case is reduced to application of Hall theorem [7, Theorem A1] (Lemma 10) and Gross theorem [9, Theorem B] (Lemma 11). In the second case, when a Hall subgroup H has even order (equivalently, odd index), the technique is absolutely different. We use the fact that H includes a Sylow 2-subgroup S of G, and so, by Lemma 5, we need to check that H and H^g are conjugate in $\langle H, H^g \rangle$ only for those g, that normalize S. Then we apply the structure of normalizers of Sylow 2-subgroups in finite simple groups obtained by A. S. Kondrat'ev (Lemma 12). This technique could be probably applied in a more general situation, For example, the following conjecture is of interest.

Conjecture 1. Subgroups of odd index are pronormal in finite simple groups.

In view of Lemma 5, Conjecture 1 holds for all finite simple groups possessing a self-normalizing Sylow 2-subgroup (for example, according to Kondrat'ev theorem (Lemma 12) in alternating groups of degree greater than 5, in orthogonal groups, and in most classes of sporadic and exceptional groups).

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